Heat Transfer and Melting of Ice in Embedded Channels
Dan Hughes
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Introduction

Dynamics of motions of ice sheets and glaciers are integral aspects of general analyses of the past and future trajectories of these potential sources of liquid water. Experimental and analytical studies of hydrodynamics and thermodynamics of flow of glaciers, including effects of meltwater on these dynamics, have been carried out for decades. The literature is enormous, and consideration of even a small fraction is beyond the scope of these notes.

Early work on the bulk motion of glaciers was presented by Nye [1953], who seems to have continued working in glaciology for quite some time, Nye [1965], Nye [1969], Nye and Mae [1972]. Early investigations into the hydrodynamics of liquid water inside of, on top of, and under masses of ice seems to have been initialed by Röthlisberger [1968, 1972]. The flow channels for melted ice are frequently labeled R-tunnels, or R-channels, after his work. R-tunnels can be huge, as in the scale of transportation tunnels for trains on rails. Initial investigations of these were driven by the occurrence of Jökulhlaups, the sudden release of massive amounts of liquid water following loss of the mechanisms initially impounding the liquid, ice dams or levitation of an ice mass above its grounding location at bedrock base, for examples. These large rapid floods can be dangerous and destructive thus driving interests in better understanding.

Sketches and photos of glaciers and other masses of ice and the various aspects of thermodynamics and hydrodynamics are readily available on the Web. Meltwater might flow along the surface like a stream or river, accumulate in surface lakes, flow downward into open crevasses or moulins, accumulate as lakes interior to the ice mass, flow as a sheet of liquid between the ice bottom and bedrock, or flow enclosed in channels partially or completely embedded within the ice mass. If the liquid drains to bedrock beneath the ice, bulk motions of the ice might be enhanced, Walder [1982], Weertman [1969], for examples.

Comprehensive reviews are given in Fountain and Walder [1998], Walder [2010], Chu [2014], Flowers [2015], Greenwood *et al*. [2016], and Nienow *et al*. [2017], among others. Chu [2014], for example, has sketches of some of the possible flow arrangements.

The focus in these notes is the steady-state energy balance equation applied to flow of liquid water in ice channels embedded in large ice masses. Several formulations of general equation systems have appeared since Röthlisberger. Development and analyses by Nye [1976], Shreve [1972], and Weertnam [1972], for examples.

The formulation by Spring and Hutter [1981, 1982] based on the dissertation by Spring [1979] is the focus in these notes. Spring's dissertation is in German, and I have not read it. The Spring-Hutter development is based on principles of continuum mechanics, and detailed mathematical reduction to the standard 1-dimensional channel-average form for engineering applications. Clarifications and modifications and applications of Spring-Hutter have been given by Clarke [2003] and Creyts and Clarke [2010]. Mayaud [2012] and Macdonald [2013] use Spring-Hutter model. Hutter [1982, 1977] has also discussed motions of glaciers and thermodynamics.

The specific systems considered here are Equations (2.1)-(2.4) in Spring and Hutter [1981], who also give a steady-state form in Equations (2.8) and (2.9), Equations (4.46)-(4.49) in Spring and Hutter [1982], with specialization to a circular channel immediately following those, Equations (7)-(20) in Clarke [2003], and Equations (3a)-(3d) in Creyts and Clarke [2010]. Spring and Hutter compare and contrast their equations with Nye [1976].

Sommers and Rajaram [2020] have developed numerical solutions for the 2-dimensional, local-instantaneous temperature equation. Results reported there agree with specific results reported herein. The formulation neglects accounting for the melted ice entering the main flow stream from the wall. As is the case for effects of viscous dissipation within the bulk liquid, the melted ice affects the gradients of temperature and velocity at the liquid-ice interface and thus the mass, momentum, and energy exchanges between liquid and ice.

Experimental validations of the model equations are generally post facto and are severely limited by the inherently complex nature of the physical domain, including complexity and hostile nature of the natural environment were the ice is located. Many important geometric details of embedded channels are simply unknowable.

Comparisons of modeling results with post facto reconstruction of Jökulhlaups have been given by Spring and Hutter [1981], and Clarke [2003], for examples. Very limited data for channel melting in a block of ice by Isenko *et al*. [2005], development of a channel following loss of an ice dam by Zou *et al*. [2022], and artificial moulins by Pohle *et al*. [2020a, 2020b].

General Equations

A summary of the general transient form of the system of interest for compressible liquid is given below for completeness, as these might be subjects for future additional notes. Generally, compressibility is neglected in almost all applications.

The Spring-Hutter system considers the case of evolution in time and space of the flow area of the channel. Changes in flow area are caused by ice melting and dynamics of the ice in which channels are located. Dynamics of the ice matrix potentially causes opening and closure, and even destruction of the channel. Those matters are not considered here.

Various channel shapes are considered in the literature, including water sheets between the ice and materials at the base of the ice. Details of the structure of channels are generally modeled by channel-average representations, such as wetted and heated perimeters, and associated equivalent, or effective, length scales. These are used in engineering correlations and mechanistic models for details that were annihilated by averaging of local-instantaneous formulations of basic equations. These matters are not considered in these notes, instead they are accepted as Standard Operating Procedures for engineering applications.

Channel Properties

The glaciology literature generally uses the Hydraulic Radius as representative channel-average length scale, which is defined as the ratio of flow area and wetted perimeter. For the case of channels flowing full and in which the ice channel fully encloses the flow, the hydraulic radius is

$$R_{hyd} = \frac{A_f}{P_{wet}}$$

Specific considerations for partially full channels, or channels not completely bounded by ice are straightforward, a semi-circular channel between the ice and the bed material, for example.

Engineering correlations generally use equivalent wetted and equivalent heated diameters, defined

$$D_{hy} = \frac{4\,A_f}{P_{wet}}$$

$$D_{he} = \frac{4\,A_f}{P_{htd}}$$

And a scale that represents the area density across which a driving potential is effective

$$\bar{A} = \frac{wall\,area}{fluid\,volume}$$

In these notes the channel will be taken to be circular with diameter D, flowing full with momentum and energy exchange occurring all round the channel. The representative dimensions are $R_{hyd} = D/4$, $D_{hy} = D_{he} = D$, and area density = $4/D$. A large area density indicates a potential for significant mass, momentum, and energy exchanges relative to the amount of material available.

For example, consider the case of cooling a tall glass of Sweet Tea on a hot humid day by pouring the Sweet Tea over ice in a glass. We'll neglect effects of the drops of condensation coalescing and sliding down the outside of the glass. The shape of the ice affects the rate of melting and cooling, with cubes, for example, having a greater surface area per unit mass of ice than, say, spherical ice globs. The melting ice also dilutes the Sweet Tea as the meltwater mixes with and cools the Tea.

The transient equations for compressible flow are given, followed by specialization to the Spring-Hutter incompressible case, and then the steady state form is summarized. The general transient compressible model potentially has interesting issues for flows in deformable channels, mathematical characteristics representing propagation speeds, and stability in the physical domain, for examples.

Nomenclature

A_f cross-sectional flow area normal to the mean flow direction,
$\bar{A}_{wf}$ wall area per unit fluid volume,
D diameter of circular channel
D_{he} heated equivalent diameter for energy exchange,
D_{hy} wetted equivalent diameter for momentum exchange,
R_{hyd} hydraulic radius,
U mean flow velocity,
z distance along the flow channel,
P_w wetted perimeter for momentum exchange
p liquid pressure,
p_i ice pressure,
$\dot{m}$ mass exchange due to melting,
$\dot{A}$ closure rate model for the ice,
ρ_l and ρ_i liquid and ice density,
Q volumetric flow of liquid

Mass Balance

Mass balance for the liquid water flowing in the channel is

$$\frac{\partial}{\partial t}\rho_l A_f + \frac{\partial}{\partial z}\rho_l U_l A_f = \dot{m}_{il} \tag{1}$$

The nomenclature is more or less standard for density, flow area, velocity, and mass exchange. The z-direction is taken along the mean flow direction in the channel and is not necessarily vertical. The subscript 'l' and 'i' refer to the liquid and ice, respectively, with the 'il' nemon referring to mass exchange between ice and liquid due to melting. The potential for refreezing of the liquid, Ozisik and Mulligan [1969], and classic Stefan phase-change problems, Alexiades and Solomon [1993], are not considered. The right-hand side is mass exchange between liquid and ice, with melting of ice into liquid taken positive. The constant density, incompressible form, the approach generally taken in glaciology is

$$\frac{\partial}{\partial t}A_f + \frac{\partial}{\partial z}U_l A_f = \frac{1}{\rho_l}\dot{m}_{il} \tag{2}$$

For numerical solution purposes, Clarke [2003] took the density to be a function of pressure in Eq. (1) only for the time derivative to get

$$\left(\frac{\partial \rho_l}{\partial p}\right)\frac{1}{\rho_l}A_f\frac{\partial}{\partial t}p + \frac{\partial}{\partial t}A_f + \frac{\partial}{\partial z}U_l A_f = \frac{1}{\rho_l}\dot{m}_{il} \tag{3}$$

For these notes the steady-state form is

$$\frac{\partial}{\partial z}U_l A_f = \frac{1}{\rho_l}\dot{m}_{il} \tag{4}$$

The volumetric flow can increase along the channel due to melting. The melting itself, by increasing the channel flow area especially at the inlet, can also increase the volumetric flow.

Flow Area Evolution

The flow area is taken to be constant for example calculations in these notes. That assumption can be checked following calculation of the melting rate and the amount of melted ice. For checkout calculations, initial and boundary conditions can always be selected to minimize melt rate.

$$\frac{\partial}{\partial t}A_f = \frac{\dot{m}}{\rho_i} - 2sgn(p_e)\left\{\frac{|p_e|}{nB}\right\}^n A_f \tag{5}$$

where the pressure difference driving changes in flow area is

$$p_e = p_i - p_l$$

The second term on the right-had side represents effects of ice dynamic on the channel. If ice pressure is greater than liquid pressure, the modeling postulates the flow area will potentially decrease. For the initial calculations in these notes, the flow area is assumed to be circular with constant diameter. This assumption does not impact the conclusions that are based on the calculations below with the energy equation.

Momentum Balance

The standard engineering channel-average, 1-dimensional formulation

$$\frac{\partial}{\partial t} U_l + U_l \frac{\partial}{\partial z} U_l = -\frac{1}{\rho_l} \frac{\partial}{\partial z} p - \frac{1}{\rho_l A_f} P_w \tau_{li} + g_z \tag{6}$$

where the impulse effects of mass exchange have been dropped. The liquid pressure distribution along the channel is generally obtained by solving the steady state momentum balance.

$$\frac{1}{\rho_l} \frac{d}{dz} p = -\frac{1}{2} \frac{d}{dz} \frac{Q^2}{A_f^2} - \frac{1}{\rho_l A_f} P_w \tau_{li} + g_z \tag{7}$$

For horizontal flows, the last term is zero. If the flow at the entrance to a channel is known, that can be specified as an initial condition for numerical integration of the mass balance Eq. (4). Given the pressure at an inlet, along with the flow, the pressure distribution can be solved as an initial value problem in space, Eq. (7). If the flow is not specified, and pressure is known at a channel outlet, the equations are set as a boundary value problem for the flow and pressure distribution. For the applications herein, the flow at an inlet is specified, and assumed constant, and the pressure distribution is not necessary. The objective is focused on effects of viscous dissipation, melting, and liquid-ice energy exchange on melting.

For plain parallel shear flows, which might not be the case in ice-channel flows, the liquid-to-ice stress is

$$\tau_{li} = \frac{1}{8} f_{li} \rho_l U_l |U_l|$$

where the friction factor is a function of the flow pattern and the geometry of the channel walls. The latter is basically unknown for ice channels in the physical domain. The Darcy-Weisback model, Brown [2002], and Gilley *et al*. [1992], and standard correlations are frequently used for example calculations. In these notes we use the Blasius correlation for large Reynolds number

$$f_{li} = 0.184 / Re^{.2}$$

This modeling is basically inappropriate for real-world ice channels and is discussed in section **Closure of the Model** below. The potential problems are well known in the glaciology literature, but the correlation is utilized for exploratory calculations, Spring and Hutter [1981, 1982], Clarke [2003], Creyts and Clarke [2010], and Sommers and Rajaram [2020].

Temperature Equation

The energy balance is expressed in terms of temporal and spatial evolution of the temperature of the liquid accounting for energy exchange between liquid and ice, viscous dissipation of kinetic energy into thermal energy addition into the bulk fluid, and effects of meltwater on the liquid temperature.

$$\rho_l C_{Pl} A_f \frac{\partial}{\partial t} T_l + \rho_l C_{Pl} A_f U_l \frac{\partial}{\partial z} T_l$$
$$= P_w \tau_0 U_l - \frac{P_w k_l Nu}{D_{he}} (T_l - T_i)$$
$$- \frac{P_w k_l Nu (T_l - T_i)}{D_{he} L_{mlt}} \rho_l C_{Pi} A_f T_i \quad (8)$$

The terms on the right-hand side are changes in liquid temperature due to:

Viscous dissipation of friction into thermal energy,
Energy transfer between the liquid and ice at the liquid-ice interface,
Melted ice mixing into the mean liquid flow. Peralta *et al*. [2025]

The ice temperature is taken to be constant, so a conduction equation and boundary conditions in the ice are not needed. Further, the ice is assumed to be at its melting temperature so accounting for sensible-energy effects is not necessary. In glaciology literature ice is frequently taken to be at its pressure-melting temperature and that is determined by the vertical depth of the ice. Liquid temperature at channel inlet is

$$T_l(z = 0) = T_{l0}$$

The classic problem to determine the final temperature of a container of liquid and ice, following melting of a quantity of ice, directly illustrates the source of accounting for meltwater effects, the last term in Eq. (8).

Steady-state temperature distribution is obtained from

$$\frac{\partial}{\partial z}T_l = \frac{P_w \tau_0}{\rho_l C_{Pl} A_f} - \frac{P_w k_l Nu}{D_{he} \rho_l C_{Pl} A_f U_l}(T_l - T_i) - \frac{P_w k_l Nu (T_l - T_i)}{D_{he} \rho_l C_{Pl} A_f U_l L_{mlt}} \rho_l C_{Pi} T_i \quad (9)$$

The second and third terms on the right-hand side are based on liquid to ice heat transfer per unit length

$$Q_{li} = \frac{P_w k_l Nu}{D_{he}}(T_l - T_i) \quad (10)$$

and with ice at constant melting temperature, mass exchange due to liquid-to-ice heat transfer is

$$\dot{m}_{il} = Q_{li}/L_{mlt} \quad (11)$$

The Nusselt number is generally calculated by the McAdams correlation, Winterton [1998], Williams [2011], or almost any convective heat transfer textbook.

$$Nu = 0.023 Re^{.8} Pr^{.4} \quad (12)$$

As in the case of the wall shear stress given above, this correlation is highly likely to be fundamentally lacking for applications in real-world ice channels. As before, see discussions in Spring and Hutter [1981, 1982], Clarke [2003], Creyts and Clarke [2010], and Sommers and Rajaram [2020], and section **Closure of the Model** below.

The last RHS term in Eq. (9) moderates effects of viscous dissipation by decreasing the liquid temperature and thus the liquid-to-ice energy exchange and melting.

For the general transient case there are four unknowns in the model, fluid velocity U_l, flow area A_f, pressure p, and liquid temperature T_l, and four equations, Equs. (1) or (3), (5), (6), and (8). If the liquid is taken to be compressible these are supplemented by an Equation of State that returns the unknown density given two thermodynamically independent thermodynamic-state properties, pressure and temperature for example.

Note that the quantity of ice that is destined to be melted, denoted by F in the Spring-Hutter [1981, 1982] development, cannot be treated as an unknown because there are not sufficient equations for the five unknowns.

The ice temperature is assumed to be constant, and the ice pressure is determined by the structure of ice above the channel. The ice temperature is frequently taken to be the pressure-melting temperature at ice pressure. We simply assume constant ice temperature and constant flow area channel to allow focus on solely viscous dissipation of kinetic energy and melting.

The transient continuous form of the general system can be investigated relative to its mathematical characteristics and linear dispersion properties. The algebraic terms in the equations sometimes lead to systems that are not strictly hyperbolic, and that gives systems for which stable numerical solutions for the discrete approximations cannot be attained. These concepts are basis for additional investigations.

Dimensionless Form for Temperature Equation

The steady-state energy equation, Eq. (9), can be written in dimensionless form as follow. Define a dimensionless temperature

$$\bar{T}_l = \frac{T_l - T_i}{T_{l0} - T_i} \tag{13a}$$

and a scaled spatial distance by

$$\bar{z} = C_{h\lambda} z \qquad (13b)$$

where

$$C_{h\lambda} = \frac{P_w k_l Nu}{D_{he} \rho_l C_{Pl} A_f U_l} \left(1 + \frac{C_{pi} T_i}{L_{mlt}}\right) \qquad (13c)$$

The factor in paratheses arises from the last right-hand side term in Eq. (9) that accounts for mixing of meltwater with the bulk liquid. The ratio of specific energy content to fusion energy is the Stefan number.

Equation (9) becomes

$$\frac{d}{d\bar{z}} \bar{T}_l + \bar{T}_l = Ec_0 \qquad (14)$$

where Ec_0 is an Eckert Number, the ratio of viscous dissipation to liquid-to-ice energy exchange, evaluated at channel inlet,

$$Ec_0 = \frac{P_w \tau_0 / \rho_l C_{Pl} A_f}{C_{h\lambda} (T_{l0} - T_i)} \qquad (15)$$

The initial condition becomes

$$\bar{T}_l (\bar{z} = 0) = 1$$

and the solution is

$$\bar{T}_l = Ec_0 (1 - e^{-\bar{z}}) + e^{-\bar{z}} \qquad (16)$$

For sufficiently long flow channels,

$$\lim_{\bar{z} \to \infty} \bar{T}_l = Ec_0 \qquad (17)$$

Note that if the last right-hand side term in Eq. (9) is omitted, the definitions and solution are the same as here, but the physical distance along the channel is different. That is, the dimensionless distance is converted to a physical distance by using Eq. (13b) without the factor in paratheses. The exponentials decay rapidly with dimensionless distance with the first term rapidly approaching 1.0 and the second term approaching 0.0. The physical distances can be quite long, and each specific application will have to determine that channels are sufficiently long to accommodate attaining the dissipation-heat-transfer balance.

Generally, for sufficiently long channels, and in absence of constraining interactions between the liquid and ice, the local liquid-to-ice heat transfer along the channel adjusts to become in balance with viscous dissipation. The conditional is equivalent to an infinite energy sink in the ice with no feedback into the liquid. Early approaches by Nye [1976] summarized by Spring and Hutter [1981, 1982], for example, simply assumed that the temperature spatial gradient was zero, indicating that the liquid temperature had sufficiently approached the ice temperature. Equations (14) and (17) indicate that the dimensionless temperature difference will approach the value represented by the modified Eckert Number at the channel inlet, including effects of meltwater on the liquid temperature. The local Eckert Number corresponding to Eq. (15)

$$Ec_0 = \frac{P_w \tau_0 / \rho_l C_{Pl} A_f}{C_{h\lambda}(T_l(z) - T_i)}$$

will approach unity for sufficiently long channels, as a plot will so indicate. If the inlet Eckert Number, Ec_0, is less than 1.0, the local value will increase along the channel. If the inlet value is greater than 1.0, the local value will decrease along the channel.

With the temperature distribution known, the liquid-to-ice heat transfer rate and associated ice melt rate distributions are available, Eq. (10) and Eq. (11). Equation (10) can be written in dimensionless form

$$\frac{Q_{li}}{Q_{li0}} = \bar{T}_l \tag{18}$$

And Eq. (11) as

$$\frac{\dot{m}_{il}}{\dot{m}_{il0}} = \bar{T}_l \tag{19}$$

Appendix A gives the results in dimensional form, along with comparisons with other solutions from the literature

Cumulative Energy and Mass exchange along the Channel

Given the spatial temperature distribution along the channel integration of the temperature along the channel, Eq. (16), gives the cumulative liquid-ice heat exchange and consequent melting. For example,

$$\begin{aligned}\mathcal{T}_{\bar{Z}_L} &= \int_0^{\bar{Z}_L} \bar{T}_l(\bar{z})\, d\bar{z} \\ &= 1 + Ec_0(\bar{Z}_L - 1) - (1 - Ec_0)\mathrm{Cosh}(\bar{Z}_L) + (1 - Ec_0)\mathrm{Sinh}(\bar{Z}_L)\end{aligned}$$

where the hyperbolic functions can be replaced by exponentials. Note that as Ec_0 is greater than or less than 1.0, the signs of the last two terms change. Note, too, that for sufficiently long channels, the cumulative distributions increase linearly with distance, second term on RHS, reflecting existence of viscous dissipation. The cumulative heat transfer and mass exchange are given by this integration and Equs. (18) and (19).

For engineered energy exchange systems, the objective is generally to maximize energy exchange between working fluid in the systems, making the heat transfer transverse to the mean flow significantly larger than viscous dissipation, Temperature changes in the direction of the mean flows are also correspondingly large. Generally, the Eckert

Number is much smaller than 1.0 for engineering heat transfer applications. In fact, entire textbooks on convective heat transfer have been written without even an introduction to the existence of the process.

In the present applications, ice melt is a primary response function of interest. Given that the ice is frequently taken to be at melting temperature, any energy available for melting might be potentially important. A focus on kinetic energy dissipation into the bulk liquid is appropriate.

Summary of the Analytical Solutions

The dimensionless temperature and cumulative solutions contain every possible solution to the steady-state temperature equation under the assumptions stated. Conversions between dimensionless and dimensional variables are carried out by use of the definitions of dimensionless valuables give above in these notes.

Closure of the Model

Averaging across the channel annihilates all details of the flow below the spatial scale of the channel flow area. Important physical phenomena and processes below this scale are represented by correlations of empirical data, mechanistic models and associated parameters. The friction factor representing the velocity gradient at the channel wall, and the heat transfer coefficient representing the temperature gradient at the wall are necessary. Applications to ice-channel flows introduces critical uncertainties into these correlations and models including even the true nature of the geometric details of the channel itself.

The balance equations require specification of a wall friction factor and heat transfer coefficient to close the system. These are generally taken from the engineering literature in which they play a critically important part. There are, however, uncertainties associated with applying this approach to ice-channel flows. As previously mentioned above, geometric details of ice channels are unknown, and basically unknowable. Uniformly smooth and symmetrical melting of channels walls would seem to be unlikely. Correlation for turbulent flows in smooth and rough engineered pipes only provide guidance. The "roughness" in pipes, generally of a microscopic scale, is unlikely to be representative of physical domain ice channels. There also are correlations that account for engineering pipe roughness, but these very likely are not directly applicable for the present applications.

Viscous energy dissipation within the bulk liquid affects gradients of fluid velocity component and temperature at the liquid-ice interface, and so affects the friction and heat transfer coefficients, Sommer and Rajaram [2020]. Melting and injection of the meltwater into the mean flow at the interface also affects the gradients and coefficients.

The details of these matters are not required for the present applications. Numerical values from the Blasius and McAdams correlations could be arbitrarily increased, multiply by 10 for example, but that leads to a washout in the end as these appear in the Eckert Number, Eq. (15), as a ratio. If one is increased, the other should also be increased. The Reynolds analogy provides some guidance relative to connection between the friction factor and Nusselt number, Sommer and Rajaram [2020].

Example Calculations

An example temperature distribution along a channel is given in the following discussion. I have presented sufficient information to allow for calculations of all kinds of distributions, and these can be carried out in a simple spreadsheet. The example here has the inlet Eckert Number less than 1.0, the usual situation.

Physical constants

Property	Value Units
Liquid density	1000. kg m^{-3}
Ice density	915. kg m^{-3}
Liquid specific heat capacity	4217.7 J $kg^{-1}K^{-1}$
Ice specific heat capacity	2108 J $kg^{-1}K^{-1}$
Liquid thermal conductivity	0.5699 W m^{-1} K^{-1}
Ice thermal conductivity	2.22 W m^{-1} K^{-1}
Ice latent heat of melting	3.33560 J kg^{-1}
Liquid viscosity	0.0016626 N s m^{-2}

Channel Conditions	
Diameter	1.0 m
Inlet velocity	4.0 m s^{-1}
Ice temperature	0.0 C
Liquid inlet temperature	1.0 C

Figures 1 and 2 are for U = 4.0 m s^{-1}, making Ec_0 less than 1.0

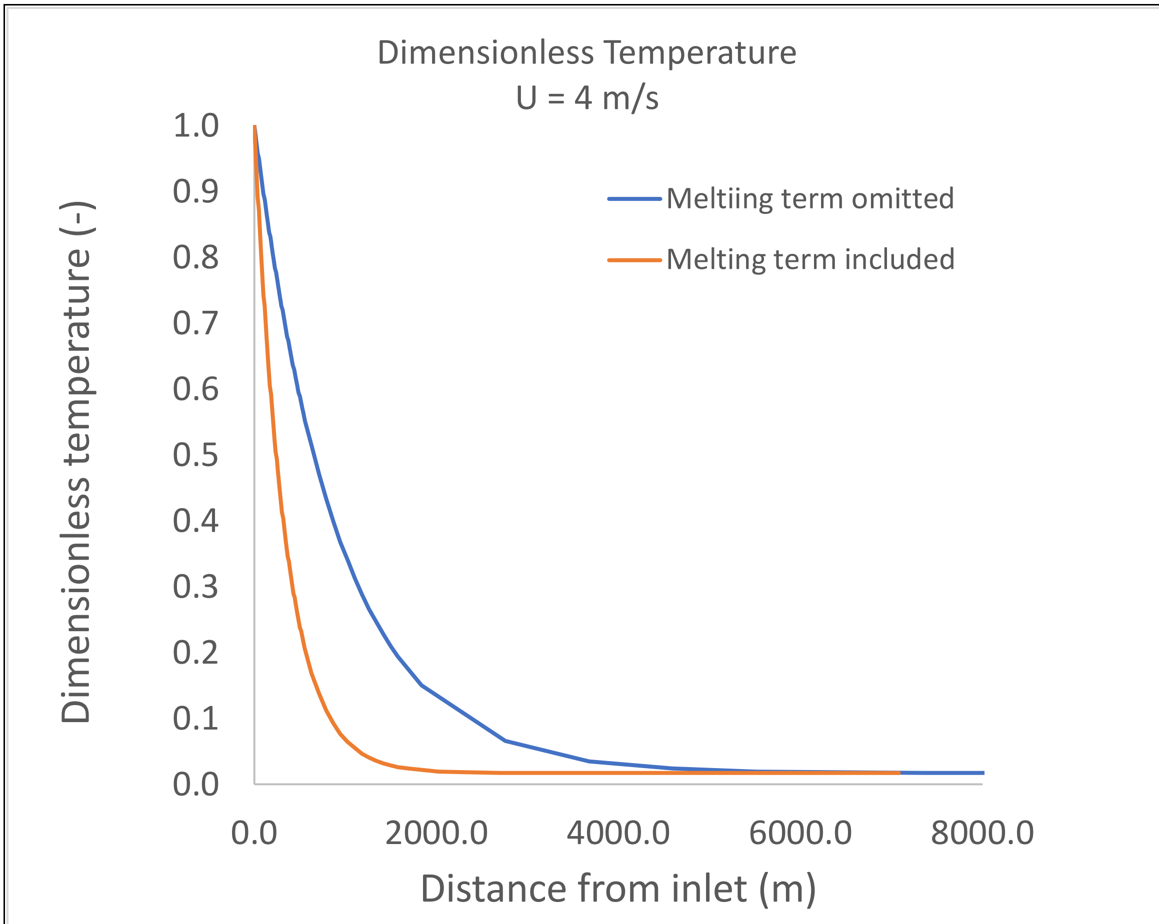

Figure 1. Dimensionless Temperature Distribution along Channel, Ec_0 less than 1.0

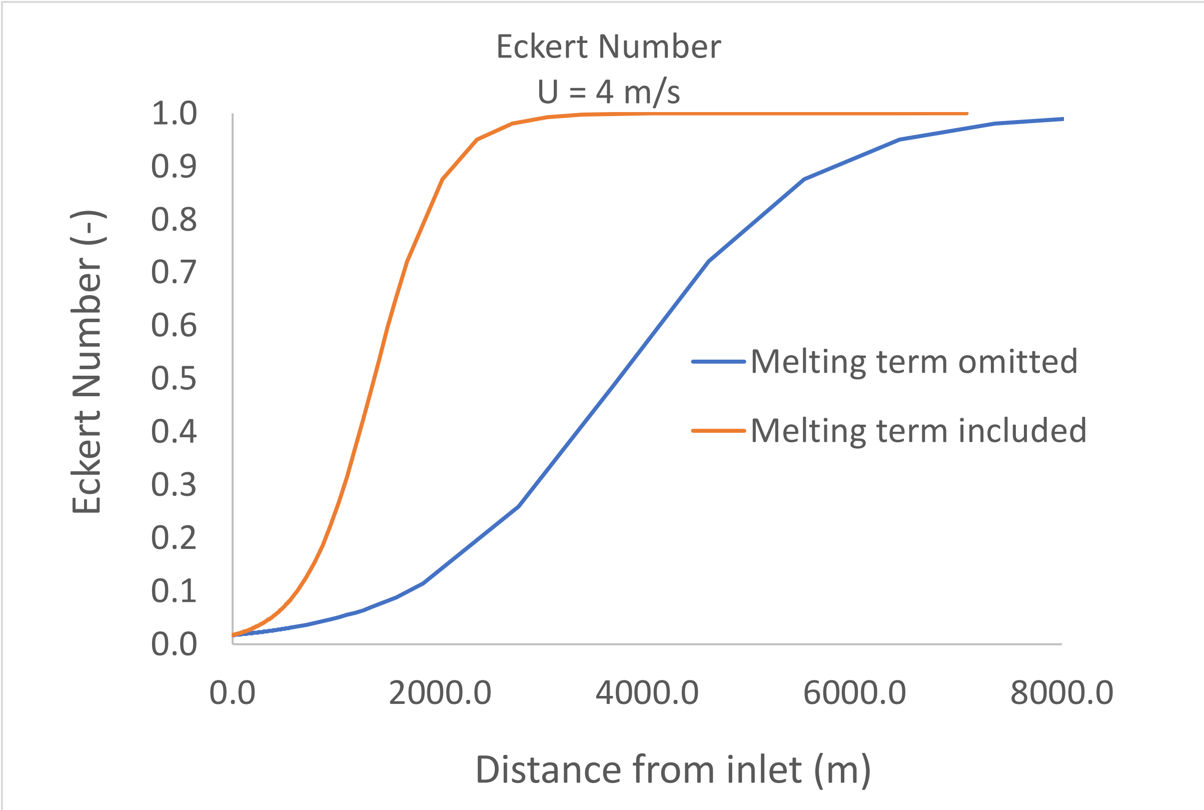

Figure 2. Eckert Number Distribution along Channel, Ec_0 less than 1.0

Figures 3 and 4 are for 40.0 m s^{-1}, making Ec_0 greater than 1.0

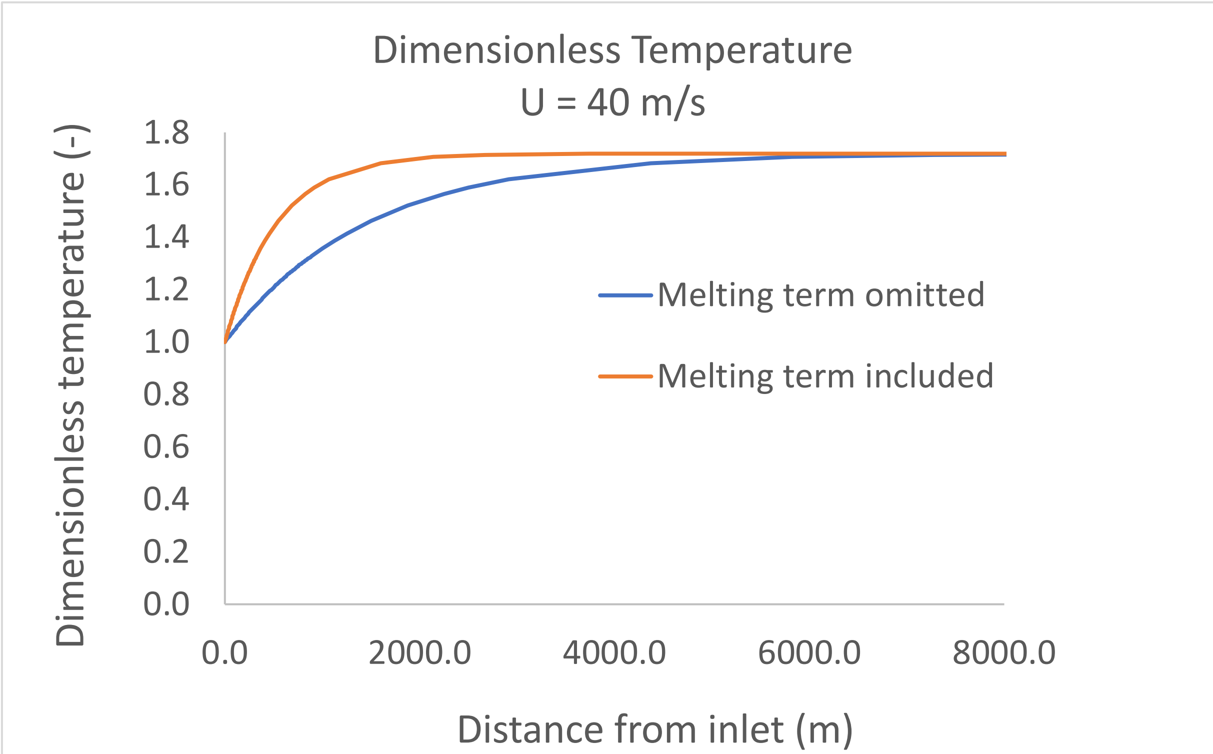

Figure 3. Dimensionless Temperature Distribution along Channel, Ec_0 greater than 1.0

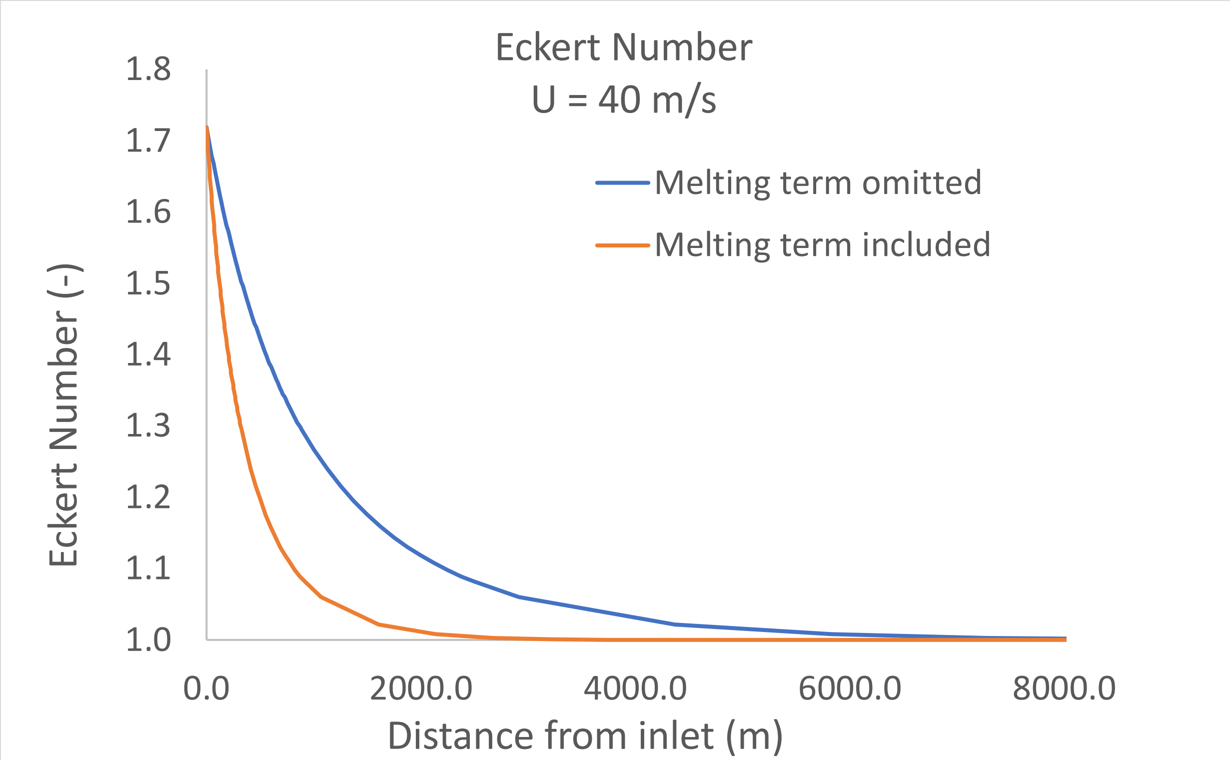

Figure 4. Eckert Number Distribution along Channel, Ec_0 greater than 1.0

Appendix A
Dimensional Solutions

Solutions for the temperature distribution are given in dimensional form in the following paragraphs.

Channel without Melting

For the steady-state, constant ice temperature, constant density, and constant channel flow area case, the energy equation is not coupled to the other equations. All thermodynamic, thermophysical, and transport properties are assumed constant. For the no-melting case, the ordinary differential equation for spatial evolution of the liquid temperature, Eq. (9) in the text gives

$$\frac{\partial}{\partial z} T_l + \frac{P_w k_l Nu}{4R_{hyd}\rho_l C_{Pl} A_f U_l}(T_l - T_i) = \frac{P_w \tau_0}{\rho_l C_{Pl} A_f} \tag{A1}$$

with

$$T_l(z=0) = T_{l0}$$

The second left-hand side term is energy exchange between the liquid and wall, with *Nu* the Nusselt Number. The right-hand side term is viscous dissipation of kinetic energy into heating the liquid. When the ice is taken to be at its melting temperature, the latter term is sometimes assumed to be directly deposited into the ice wall and produce melting. The equation and the physical domain indicate that the energy instead is deposited directly into the bulk liquid. When the ice temperature is less that the melting temperature the energy exchange serves to increase the ice temperature, potentially to the melting temperature. One could argue that when the liquid is above the ice temperature and the ice temperature is less than melting, the liquid-ice interface might be at the melting temperature. We are not interested in that case.

The temperature equation shows that as the temperature gradient approaches zero, as assumed by Nye [1976], viscous dissipation is balanced by wall energy exchange. Solution with liquid temperature the specified inlet liquid temperature is

$$T_l = T_i + (T_{l0} - T_i)\, e^{-C_h z} + \frac{C_\varepsilon}{C_h}\left(1 - e^{-C_h z}\right) \tag{A2}$$

where

$$C_\varepsilon = \frac{P_w \tau_0}{\rho_l C_{Pl} A_f}$$

$$C_h = \frac{P_w k_l Nu}{4R_{hyd}\, \rho_l C_{Pl} A_f U_l} \tag{A3}$$

At very long distances from the inlet the temperature difference approaches

$$\lim_{z \to \infty} (T_l - T_i) = \frac{C_\varepsilon}{C_h}$$

Which basically comes down to a ratio between dissipation and the heat transfer coefficient, a kind of Eckert Number, *Ec*. While the dissipation is supplying energy to the fluid all along the channel, energy transfer into the ice acts to reduce the increase that the liquid experiences.

Channel with Melting

For the case of melting, the steady state liquid temperature distribution is solution of

$$\begin{aligned}&\frac{d}{dz} T_l + \frac{P_w k_l Nu}{D_{he} \rho_l C_{Pl} A_f U_l} (T_l - T_i) \\ &+ \frac{P_w k_l Nu}{D_{he} \rho_l C_{Pl} A_f U_l L_{mlt}} (T_l - T_i) C_{Pi} T_i = \frac{P_w \tau_0}{\rho_l C_{Pl} A_f}\end{aligned} \tag{A4}$$

The terms on the left-hand side, starting with the second, are changes in water temperature due to: (1) energy transfer between liquid and ice, and (2) melted ice mixing into the mean liquid flow from the channel wall. The ice temperature is assumed to be constant at the melting temperature. Note that this term is simply the wall energy exchange divided by the latent energy of fusion/melting, and the melted ice supplies energy to the bulk liquid at the ice temperature. The right-hand side is viscous dissipation of kinetic energy into heat.

Solution with melting is

$$T_l = T_i + (T_{l0} - T_i)\, e^{-C_{h\lambda} z} + \frac{C_\varepsilon}{C_{h\lambda}}(1 - e^{-C_{h\lambda} z}) \tag{A5}$$

where the coefficient including meltwater effects is

$$C_{h\lambda} = \frac{P_w\, k_l Nu}{4 R_{hyd}\, \rho_l C_{Pl} A_f U_l}\left(1 + \frac{C_{pi} T_i}{L_{mlt}}\right) \tag{A6}$$

For very long channels, the liquid-ice temperature difference approaches

$$\lim_{z \to \infty} (T_l - T_i) = \frac{C_\varepsilon}{C_{h\lambda}}$$

The melted ice mixes with the mean liquid and so reduces the driving potential for liquid-ice energy exchange.

Given solution for the spatial distribution for temperature, the liquid-ice energy exchange and associated mass exchange can be determined.

$$Q_{li}(z) = \frac{P_w k}{4R_{hyd}} Nu\left(T_l(z) - T_i\right)$$

$$\dot{m}(z) = \frac{Q_{li}(z)}{L_{mit}}$$

where *Nu* is the Nusselt number for convective heat transfer, generally given by an algebraic correlation of experimental data.

Comparison with Sommers and Rajaram [2020] Solution

Sommers and Rajaram [2020] develop a 1-D channel-averaged temperature balance in their Equation [21]. That equation does not contain accounting of effects of meltwater on the bulk liquid temperature. It is equivalent to Eq. (A1) above. Solutions are given in Section 4 in Equation [29], which is equivalent to Eq. (A2) above. The dimensionless solution Eq. (16) in the main text above clearly shows that the dimensionless liquid-to-ice temperature difference approaches the Eckert Number for sufficiently long channels.

Without accounting for meltwater effects, the parametric variations given in Figures 11 and 12 in the paper are incomplete. The "thermal entry length" concept discussed in the Sommers and Rajaram paper is additionally shortened by meltwater effects on the bulk liquid temperature. This effect is apparent from our Eqs. (A3) and (A6).

Sommers and Rajaram give a discussion of effects of channel-geometry details in Section 4.2

Comparison with Solution of Spring-Hutter Energy Equation

The original Spring-Hutter [1981, 1982] temperature equation is

$$\frac{d}{dz}T_l + \frac{P_w k_l Nu}{D_{he}\rho_l C_{Pl} A_f U_l}(T_l - T_i) + \frac{P_w k_l Nu}{D_{he}\rho_l C_{Pl} A_f U_l L_{mlt}} C_{Pi}(T_l - T_i)^2 = \frac{P_w \tau_0}{\rho_l C_{Pl} A_f} \tag{A7}$$

This formulation is Equation [2.3] and [2.8] in Spring and Hutter [1981]. Equation [4.48], [4.51], [5.39], and [6.4] in Spring and Hutter [1982]. The equation is explicitly carried over by Clarke [2003], Equations [10] and [20], and Creyts and Clarke [2010], Equations [3d] and [8a/8b].

Note the important distinction in the last left-hand side term from the formulation used in these notes. Meltwater effects are measured by the difference between the bulk liquid and ice temperature, and not by the ice temperature alone. The former approach is not correct, while the latter is.

Equation (A7) can be written in dimless form as in the main text. The analytical solution is

$$\bar{T}_l = -\left(\frac{1}{2C_{SH}}\right) \left\{1 + i\sqrt{1+4C_{SH}Ec_0}\tan\left[\frac{1}{2}i\sqrt{1+4C_{SH}Ec_0}\bar{Z} - \operatorname{atan}\left(\frac{1+2C_{SH}}{i\sqrt{1+4C_{SH}Ec_0}}\right)\right]\right\}$$

where

$$C_{SH} = C_{pl}(T_{l0} - T_i)/L_{mlt}$$

It’s not clear what the asymptotic solution goes to, but a few calculations indicated that the limit is real and approaches values that are less than the inlet Eckert Number. That is not the expected results.

Appendix B
More on the Eckert Number

An approximate value for the liquid speed that will give an Eckert Number greater than unity is developed. The development is based on the Reynolds Analogy connecting momentum and energy exchange between a fluid and a stationary surface.
https://thermopedia.com/content/1092/

The Reynolds Analogy is exact only for the case of laminar flows for which a pressure gradient is not present. This restriction is due to the lack of a term in the energy equation that corresponds to the pressure gradient in the momentum equation. The analogy is extended to turbulent single-phase flows by an argument relating to the universal smallest dissipative scales near walls in these flows, the Kolmogorov microscales.

[My Equation Editor seems to be stuck in Bold and I can't find it.]

The basic definition of the Eckert Number is

$$Ec = \frac{\varepsilon}{q_w} \tag{B1}$$

where the dissipation

$$\varepsilon = \frac{1}{2}\rho_l f_{wf} U^3 \tag{B2}$$

based on the Fanning wall friction factor. The Fanning friction factor is one-fourth the Darcy-Weisbach factor

$f_{wf} = C_f/4.$

Wall heat transfer is

$$q_w = h_c(T_f - T_w) \tag{B3}$$

With the convective heat transfer coefficient given by

$$h_c = kNu/D \tag{B4}$$

The Reynolds Analogy relates the Reynolds Number, Prandtl Number, and Nusselt Number

$$Re\frac{f_{wf}}{2} = NuPr^{-1/3} \quad (0.6 < Pr < 60) \tag{B5}$$

an extension to the original formulation.

Combining Equs. (B1) – (B5) an approximate condition that the Eckert Number be less than 1.0 is

$$U < \sqrt{C_{pl}(T_l - T_i)}\,/Pr^{2/3}$$

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